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MOMENTS OF HYPERGEOMETRIC HURWITZ ZETA FUNCTIONS

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Abstract

This paper investigates moments of hypergeometric Hurwitz zeta functions.

1. Introduction

In this paper, we investigate moments of hypergeometric Hurwitz zeta functions defined by

$$\zeta_N(s, a) = \frac{1}{\Gamma(s+N-1)} \int_0^\infty \frac{x^{s+N-2} e^{(1-a)x}}{e^x - T_{N-1}(x)}\, dx,$$

$$N \geq 1, \quad 0 < a \leq 1. \tag{1.1}$$

Throughout this paper, we assume that N is a positive integer and a is a real number with $0 < a < 1$. Observe that $\zeta_1(s, a) = \zeta(s, a)$, where $\zeta(s, a)$ is the classical Hurwitz zeta function. Following Riemann, we

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develop their analytic continuation to the entire complex plane, except for N simple poles at $s = 1, 0, -1, \cdots, 2 - N$, and establish many properties analogous to those satisfied by Riemann's zeta function.

In Section 2, we define hypergeometric zeta functions, establish convergence on a right half-plane, and develop their series representations. In Section 3, we reveal their analytic continuation to the entire complex plane, except at a finite number of poles, and calculate their residues in terms of generalized Bernoulli numbers. In Section 4, we establish a series formula valid on a left half-plane and use it to establish formulas involving moments of hypergeometric Hurwitz zeta functions.

2. Preliminaries

In this section, we formally define hypergeometric zeta functions, establish a domain of convergence, and demonstrate their series representations.

Definition 2.1. Denote the Maclaurin (Taylor) polynomial of the exponential function e^x by

$$T_N(x) = \sum_{k=0}^{N} \frac{x^k}{k!}.$$

We define the N^{th}-*order hypergeometric Hurwitz zeta function* (or just *hypergeometric Hurwitz zeta function* for short) to be

$$\zeta_N(s, a) = \frac{1}{\Gamma(s + N - 1)} \int_0^\infty \frac{x^{s+N-2} e^{(1-a)x}}{e^x - T_{N-1}(x)}\, dx,$$

$$N \geq 1, \quad 0 < a \leq 1. \tag{2.1}$$

Lemma 2.1. $\zeta_N(s, a)$ *converges absolutely for* $\sigma = \Re(s) > 1$.

Proof. Choose $0 < \alpha < 1$ such that $a + \alpha > 1$. Let $K > 0$ be such that $e^{x} \geq e^{\alpha x} + T_{N-1}(x)$ for all $x \geq K$. This is equivalent to $e^{x} - T_{N-1}(x) \geq e^{\alpha x}$. For $\sigma > 1$, we have

$$|\zeta_N(s, a)| \leq \frac{1}{|\Gamma(s+N-1)|}\left[\int_0^K \left|\frac{e^{(1-a)x}x^{s+N-2}}{e^{x} - T_{N-1}(x)}\right| dx + \int_K^\infty \left|\frac{e^{(1-a)x}x^{s+N-2}}{e^{x} - T_{N-1}(x)}\right| dx\right]$$

$$\leq \frac{1}{|\Gamma(s+N-1)|}\left[\int_0^K \frac{e^{(1-a)x}x^{\sigma+N-2}}{x^N / N!} dx + \int_K^\infty x^{\sigma+N-2}e^{(1-a-\alpha)x} dx\right].$$

The first integral is finite and since $1 - a - \alpha < 0$, the second integral is convergent. This proves our lemma. □

We establish a "Dirichlet type" series representation for hypergeometric Hurwitz zeta in the next few lemmas.

Lemma 2.2. *For* $\sigma > 1$, *we have*

$$\zeta_N(s, a) = \sum_{n=0}^{\infty} f_n(N, s, a), \tag{2.2}$$

where

$$f_n(N, s, a) = \frac{1}{\Gamma(s+N-1)}\int_0^\infty x^{s+N-2}T_{N-1}^{n}(x)e^{-(n+a)x} dx. \tag{2.3}$$

Proof. Since $|T_{N-1}(x)e^{-x}| < 1$ for all $x > 0$, we can rewrite the integrand in (2.1) as a geometric series

$$\frac{x^{s+N-2}e^{(1-a)x}}{e^{x} - T_{N-1}(x)} = \frac{e^{-ax}x^{s+N-2}}{1 - T_{N-1}(x)e^{-x}}$$

$$= e^{-ax}x^{s+N-2}\sum_{n=0}^{\infty}[T_{N-1}(x)e^{-x}]^n$$

$$= x^{s+N-2}\sum_{n=0}^{\infty}T_{N-1}^{n}(x)e^{-(n+a)x}.$$

The lemma now follows by reversing the order of integration and summation because of dominated convergence theorem

$$\zeta_N(s, a) = \frac{1}{\Gamma(s+N-1)} \int_0^\infty x^{s+N-2} \sum_{n=0}^\infty T_{N-1}^n(x) e^{-(n+a)x} dx$$

$$= \sum_{n=0}^\infty \left[\frac{1}{\Gamma(s+N-1)} \int_0^\infty x^{s+N-2} T_{N-1}^n(x) e^{-(n+a)x} dx \right].$$

□

Lemma 2.3. *For* $f_n(N, s, a)$ *given by* (2.3), *we have*

$$f_n(N, 1, a) = \frac{1}{n+a}. \tag{2.4}$$

Proof. Since $x^{N-1} = (N-1)! \, [T_{N-1}(x) - T_{N-2}(x)]$, it follows that

$$f_n(N, 1, a) = \frac{1}{(N-1)!} \int_0^\infty x^{N-1} T_{N-1}^n(x) e^{-(n+a)x} dx$$

$$= \int_0^\infty T_{N-1}^{n+1}(x) e^{-(n+a)x} dx - \int_0^\infty T_{N-2}(x) T_{N-1}^n(x) e^{-(n+a)x} dx.$$

But the two integrals above merely differ by $1/(n+a)$, which results from integrating by parts

$$\int_0^\infty T_{N-1}^{n+1}(x) e^{-(n+a)x} dx = \frac{1}{n+a} + \int_0^\infty T_{N-2}(x) T_{N-1}^n(x) e^{-(n+a)x} dx.$$

This establishes the lemma. □

Lemma 2.4. *For* $\Re(s) = \sigma > 1$, *we have*

$$\zeta_N(s, a) = \sum_{n=0}^\infty \frac{\mu_N(n, s, a)}{(n+a)^{s+N-1}}, \tag{2.5}$$

where

$$\mu_N(n, s, a) = \sum_{k=0}^{n(N-1)} \frac{a_k(N, n)}{(n+a)^k} (s+N-1)_k. \tag{2.6}$$

Here $a_k(N, n)$ *is generated by*

$$(T_{N-1}(x))^n = \left(\sum_{k=0}^{N-1} \frac{x^k}{k!} \right)^n = \sum_{k=0}^{n(N-1)} a_k(N, n) x^k.$$

Proof. With $a_k(N, n)$ as given above, we have

$$\zeta_N(s, a) = \sum_{n=0}^{\infty} \frac{1}{\Gamma(s+N-1)} \int_0^\infty x^{s+N-2} T_{N-1}^n(x) e^{-(n+a)x} dx$$

$$= \sum_{n=0}^{\infty} \left[\frac{1}{\Gamma(s+N-1)} \int_0^\infty x^{s+N-2} \left(\sum_{k=0}^{n(N-1)} a_k(N, n) x^k \right) e^{-(n+a)x} dx \right]$$

$$= \sum_{n=0}^{\infty} \left[\frac{1}{\Gamma(s+N-1)} \frac{1}{n^{s+N-1}} \int_0^\infty \left(\sum_{k=0}^{n(N-1)} a_k(N, n) \frac{x^{s+k+N-2}}{(n+a)^k} \right) e^{-x} dx \right]$$

$$= \sum_{n=0}^{\infty} \left[\frac{1}{\Gamma(s+N-1)} \frac{1}{n^{s+N-1}} \left(\sum_{k=0}^{n(N-1)} \frac{a_k(N, n)}{(n+a)^k} \int_0^\infty x^{s+k+N-2} e^{-x} dx \right) \right]$$

$$= \sum_{n=0}^{\infty} \left[\frac{1}{n^{s+N-1}} \left(\sum_{k=0}^{n(N-1)} \frac{a_k(N, n)}{(n+a)^k} \frac{\Gamma(s+N+k-1)}{\Gamma(s+N-1)} \right) \right]$$

$$= \sum_{n=0}^{\infty} \left[\frac{1}{(n+a)^{s+N-1}} \left(\sum_{k=0}^{n(N-1)} \frac{a_k(N, n)}{(n+a)^k} (s+N-1)_k \right) \right]$$

$$= \sum_{n=0}^{\infty} \frac{\mu_N(n, s, a)}{(n+a)^{s+N-1}}.$$

$\square$

Lemma 2.5.

$$\mu_N(n, 1, a) = \sum_{k=0}^{n(N-1)} \frac{a_k(N, n)}{(n+a)^k}(N)_k = (n+a)^{N-1}. \tag{2.7}$$

Proof. Since $\mu_N(n, s, a)/(n+a)^{s+N-1} = f_n(N, s, a)$, we have from (2.4) that $\mu_N(n, 1, a)/(n+a)^N = f_n(N, 1, a) = 1/(n+a)$. The result of the lemma now becomes clear. □

3. Analytic Continuation

In this section, we follow Riemann by using contour integration to develop the analytic continuation. To this end, consider the contour integral

$$I_N(s, a) = \frac{1}{2\pi i}\int_\gamma \frac{(-w)^{s+N-1}e^{(1-a)w}}{e^w - T_{N-1}(w)}\frac{dw}{w}, \tag{3.1}$$

where the contour γ is taken to be along the real axis from ∞ to $\delta > 0$, then counterclockwise around the circle of radius δ, and lastly along the real axis from δ to ∞ (cf. Figure 1). Moreover, we let $-w$ have argument $-\pi$ backwards along ∞ to δ and argument π when going to ∞. Also, we choose the radius δ to be sufficiently small (depending on N) so that, there are no roots of $e^w - T_{N-1}(w) = 0$ inside the circle of radius δ besides the trivial root $z_0 = 0$. This follows from the fact that $z_0 = 0$ is an isolated zero. It is then clear from this assumption that $I_N(s, a)$ must converge for all complex s and therefore defines an entire function.

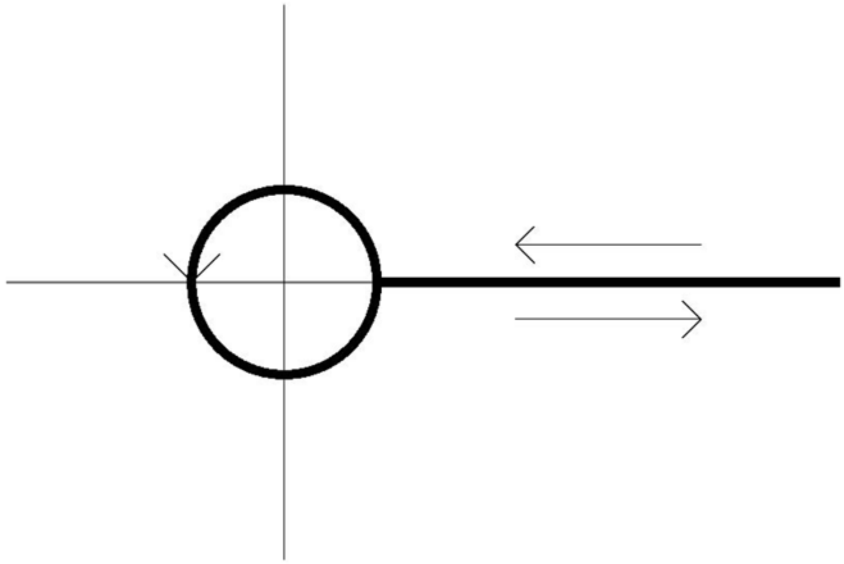

Figure 1. Contour γ.

Remark 3.1. (a) To be precise the contour γ should be taken as a limit of contours γ_ϵ as $\epsilon \to 0$, where the portions running along the x-axis are positioned at heights $\pm\epsilon$. Moreover, the poles of the integrand in (3.1) cannot accumulate inside this strip due to the asymptotic exponential growth of the zeros of $e^w - T_{N-1}(w) = 0$ (see [4]).

(b) Since we are most interested in the properties of $I_N(s, a)$ in the limiting case when $\delta \to 0$, we will also write $I_N(s, a)$ to denote $\lim_{\delta\to 0} I_N(s, a)$.

We begin by evaluating $I_N(s, a)$ at integer values of s. To this end, we decompose it as follows:

$$I_N(s, a) = \frac{1}{2\pi i}\int_\infty^\delta \frac{e^{(1-a)x} e^{(s+N-1)(\log x - \pi i)}}{e^x - T_{N-1}(x)} \frac{dx}{x}$$

$$+\frac{1}{2\pi i}\int_{|w|=\delta}\frac{(-w)^{s+N-1}e^{(1-a)w}}{e^{w}-T_{N-1}(w)}\frac{dw}{w}$$

$$+\frac{1}{2\pi i}\int_{\delta}^{\infty}\frac{e^{(1-a)x}e^{(s+N-1)(\log x+i\pi)}}{e^{x}-T_{N-1}(x)}\frac{dx}{x}. \tag{3.2}$$

Now, for integer $s = n$, the two integrations along the real axis in (3.2) cancel and we are left with just the middle integral around the circle of radius δ

$$I_N(n, a) = \frac{1}{2\pi i}\int_{|w|=\delta}\frac{(-w)^{n+N-1}e^{(1-a)w}}{e^{w}-T_{N-1}(w)}\frac{dw}{w}.$$

Since the expression $e^{(1-a)w}w^N(e^w - T_{N-1}(w))^{-1}$ inside the integrand has a removable singularity at the origin, it follows by Cauchy's theorem that for integers $n > 1$,

$$I_N(n, a) = 0.$$

For integers $n \leq 1$, we consider the power series expansion

$$\frac{w^N e^{(1-a)w}/N!}{e^{w}-T_{N-1}(w)} = \sum_{m=0}^{\infty}\frac{B_{N,m}(1-a)}{m!}w^m. \tag{3.3}$$

It now follows from the residue theorem that

$$I_N(n, a) = \frac{1}{2\pi i}\int_{|w|=\delta}\frac{(-w)^{n+N-1}e^{(1-a)w}}{e^{w}-T_{N-1}(w)}\frac{dw}{w}$$

$$= (-1)^{n+N-1}\frac{N!}{2\pi i}\int_{|w|=\delta}\left(\sum_{m=0}^{\infty}\frac{B_{N,m}(1-a)}{m!}w^m\right)\frac{dw}{w^{2-n}}$$

$$= \frac{(-1)^{n+N-1}N!\,B_{N,1-n}(1-a)}{(1-n)!}. \tag{3.4}$$

We now express $\zeta_N(s, a)$ in terms of $I_N(s, a)$. For $\Re(s) = \sigma > 1$, the middle integral in (3.2) goes to zero as $\delta \to 0$. It follows that

$$I_N(s, a) = \left(\frac{e^{\pi i(s+N-1)} - e^{-\pi i(s+N-1)}}{2\pi i}\right)\int_0^{\infty}(e^x - T_{N-1}(x))^{-1}e^{(1-a)x}x^{s+N-2}dx$$

$$= \frac{\sin[\pi(s + N - 1)]}{\pi}\Gamma(s + N - 1)\zeta_N(s, a).$$

Now, by using the functional equation for the gamma function

$$\Gamma(1 - (s + N - 1))\Gamma(s + N - 1) = \frac{\pi}{\sin[\pi(s + N - 1)]},$$

we obtain

$$\zeta_N(s, a) = \Gamma(1 - (s + N - 1))I_N(s, a). \tag{3.5}$$

Remark 3.2. Equation (3.5) and the fact that $0 < a < 1$ imply that the zeros of $I_N(s, a)$ at positive integers $n > 1$ are simple, since we know by Equation (1.1) that $\zeta_N(n, a) > 1$ for $n > 1$.

We close this section by proving the following theorem:

Theorem 3.1. $\zeta_N(s, a)$ *is analytic on the entire complex plane except for simple poles at* $\{2 - N, 3 - N, \cdots, 1\}$, *whose residues are*

$$Res\,(\zeta_N(s, a),\ s = n) = (2 - n)\binom{N}{2 - n}B_{N,1-n}(1 - a), \quad 2 - N \le n \le 1. \tag{3.6}$$

Furthermore, for negative integers n less than $2 - N$, *we have*

$$\zeta_N(n) = (-1)^{-n-N+1}\binom{1 - n}{N}^{-1}B_{N,1-n}(1 - a). \tag{3.7}$$

Proof. Since $\Gamma(1-(s+N-1))$ has only simple poles at $s = 2-N$, $3-N, \cdots$, and $I_N(s, a)$ has simple zeros at $s = 2, 3, \cdots$, it follows from (3.5) that $\zeta_N(s, a)$ is analytic on the whole plane except for simple poles at $s = n$, $2 - N \leq n \leq 1$. Recalling the fact that the residue of $\Gamma(s)$ at negative integer n is $(-1)^n / |n|!$, it follows from (3.4) that

$$\begin{aligned}
\operatorname{Res}(\zeta_N(s, a), s = n) &= \lim_{s \to n}(s-n)\zeta_N(s, a) \\
&= \lim_{s \to n}[(s-n)\Gamma(1-(s+N-1))I_N(s, a)] \\
&= -\frac{(-1)^{2-N-n}}{(2-N-n)!} I_N(n, a) \\
&= -\frac{(-1)^{2-N-n}}{(2-N-n)!} \frac{(-1)^{n+N-1} N!\, B_{N,1-n}(1-a)}{(1-n)!} \\
&= (2-n)\binom{N}{2-n} B_{N,1-n}(1-a),
\end{aligned}$$

which proves (3.6). For $n < 2 - N$, (3.4), (3.5), and the fact that $\Gamma(1-(n+N-1)) = (1-N-n)!$ imply

$$\begin{aligned}
\zeta_N(n, a) &= \Gamma(1-(n+N-1)I_N(n, a) \\
&= \frac{(-1)^{n+N-1} N!\,(1-N-n)!\, B_{N,1-n}(1-a)}{(1-n)!} \\
&= (-1)^{-n-N+1}\binom{1-n}{N}^{-1} B_{N,1-n}(1-a),
\end{aligned}$$

which is (3.7) . This completes the proof of the theorem. □

4. Moments of Hypergeometric Hurwitz Zeta Functions

In the present section, we discuss a 'pre-functional equation' satisfied by $\zeta_N(s, a)$. Let γ_R be the contour shown in Figure 2, where the outer circular region is part of a circle of radius $R = (2M + 1)\pi$ (M is a positive integer so that the poles of the integrand are not on the contour), the inner circle has radius $\delta < 1$, the vertical line is $\Re(z) = -1$. The outer semi-circle is traversed clockwise, the imaginary axis is traversed from bottom to top, the inner circle counterclockwise, and the radial segment along the positive real axis is traversed in both directions. Then define

$$I_{\gamma_R}(s, a) = \frac{1}{2\pi i} \int_{\gamma_R} \frac{(-z)^{s+N-1} e^{(1-a)z}}{e^z - T_{N-1}(z)} \frac{dz}{z}. \tag{4.1}$$

We claim that $I_{\gamma_R}(s, a)$ converges to $I_N(s, a)$ as $R \to \infty$ for $\Re(s) < 0$. To prove this, observe that the portion of $I_{\gamma_R}(s, a)$ around the outer circle and the imaginary axis tends to zero as $R \to \infty$ on the same domain. To prove this, we first choose a constant $P > 0$, such that

$$A|z|^{N-1} \le |T_{N-1}(z)| \le B|z|^{N-1}, \text{ for all } |z| > P.$$

But then on the outer circle defined by $|z| = |R(\cos\theta + i\sin\theta)| = (2M + 1)\pi$, if we choose $R > P$, then

$$|e^z - T_{N-1}| \ge \eta(e^x - C|z|^{N-1}),$$

where $\eta = \pm 1$ and $C = A$ or B. Thus,

$$\left|\frac{z^{N-1}e^{(1-a)z}}{e^z - T_{N-1}(z)}\right| \le \frac{e^{(1-a)x}|z|^{N-1}}{\eta(e^x - C|z|^{N-1})} = \frac{R^{N-1}e^{-aR\cos\theta}}{\eta(1 - CR^{N-1}e^{-R\cos\theta})},$$

converges to zero as $R \to \infty$, since $-\pi/2 < \theta < \pi/2$. On the imaginary axis $z = iy$, we have

$$\left|\frac{z^{N-1}e^{(1-a)z}}{e^z - T_{N-1}(z)}\right| \leq \frac{|y|^{N-1}}{A|y|^{N-1} - 1},$$

which converges to $1/A$ as $|y| \to \infty$. Since $|(-z)^s / z| < |z|^{\Re(s)-1}$ and $\Re(s) < 0$, we conclude that the integrals on the outer circle and the vertical lines both converge to zero as $R \to \infty$. On the circle of radius 2δ, the integrand is bounded and hence the integral vanishes as $\delta \to 0$.

$$I_N(s, a) = \lim_{R \to \infty} I_{\gamma_R}(s, a). \tag{4.2}$$

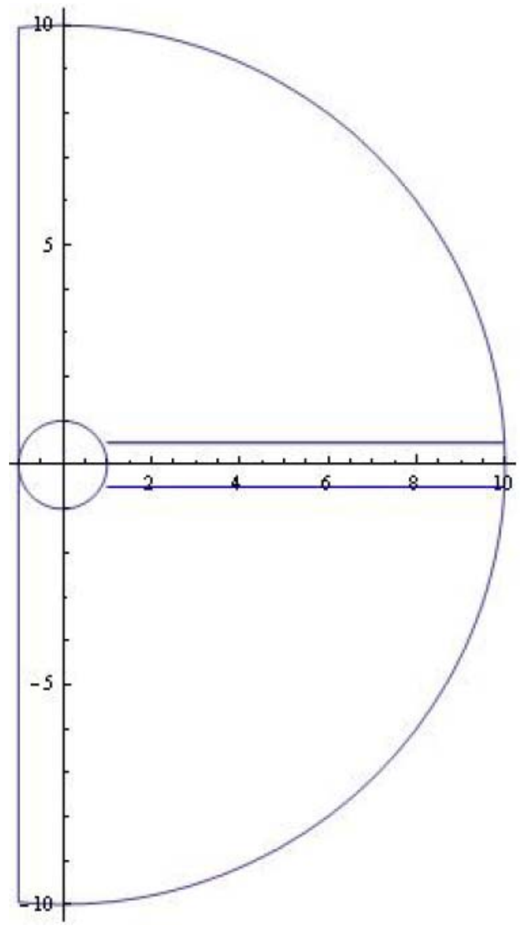

Figure 2. Contour γ_M.

On the other hand, we have by residue theory

$$I_{\gamma_R}(s, a) = -\sum_{k=1}^{K}\left[\operatorname{Res}\left(\frac{(-z)^{s+N-2}e^{(1-a)z}}{e^z - T_{N-1}(z)}, z = z_k\right)\right.$$

$$\left. + \operatorname{Res}\left(\frac{(-z)^{s+N-2}e^{(1-a)z}}{e^z - T_{N-1}(z)}, z = \bar{z}_k\right)\right]. \tag{4.3}$$

Here, $z_k = r_k e^{i\theta_k}$ and $\bar{z}_k = r_k e^{-i\theta_k}$ are the complex conjugate roots of $e^z - T_{N-1}(z) = 0$ and $K = K_M$ is the number of roots inside γ_R in the upper-half plane. Clearly, z_k depends on N. We will make this assumption throughout and use the same notation z_k instead of the more cumbersome notation $z_k(N)$. Moreover, we arrange the roots in ascending order so that $|z_1| < |z_2| < |z_3| < \cdots$, since none of the roots can have the same length (see [4]). Now, to evaluate the residues, we call upon Cauchy's integral formula

$$\operatorname{Res}\left(\frac{(-z)^{s+N-2}e^{(1-a)z}}{e^z - T_{N-1}(z)}, z = z_k\right) = (-z_k)^{s+N-2}e^{(1-a)z_k}\lim_{z\to z_k}\frac{z - z_k}{e^z - T_{N-1}(z)}.$$

Here, C_k is any sufficiently small contour enclosing only one root z_k of $e^z - T_{N-1}(z) = 0$. But then

$$\lim_{z\to z_k}\frac{z - z_k}{e^z - T_{N-1}(z)} = \frac{1}{e^{z_k} - T_{N-2}(z_k)} = \frac{(N-1)!}{z_k^{N-1}}.$$

It follows that

$$\operatorname{Res}\left(\frac{(-z)^{s+N-2}e^{(1-a)z}}{e^z - T_{N-1}(z)}, z = z_k\right) = (-1)^{N-1}(N-1)!\,(-z_k)^{s-1}e^{(1-a)z_k}.$$

Therefore,

$$I_{\gamma_R}(s, a) = (-1)^{N-1}(N-1)!\sum_{k=1}^{K}\left[(-z_k)^{s-1}e^{(1-a)z_k} + (-\bar{z}_k)^{s-1}e^{(1-a)\bar{z}_k}\right]. \tag{4.4}$$

Since $K \to \infty$ as $R \to \infty$, we have by (4.2) and (4.4),

$$I_N(s, a) = \lim_{R\to\infty} I_{\gamma_R}(s, a)$$

$$= 2(-1)^{N-1}(N-1)! \sum_{k=1}^{\infty}\left[(-z_k)^{s-1}e^{(1-a)z_k} + (-\bar{z}_k)^{s-1}e^{(1-a)\bar{z}_k}\right]. \quad (4.5)$$

Combining (3.5) and (4.5), we have proved

Theorem 4.1. *For* $\Re(s) < 0$,

$$\zeta_N(s, a) = 2(-1)^{N-1}(N-1)!\,\Gamma(1-(s+N-1))$$

$$\times \sum_{k=1}^{\infty}\left[(-z_k)^{s-1}e^{(1-a)z_k} + (-\bar{z}_k)^{s-1}e^{(1-a)\bar{z}_k}\right]. \quad (4.6)$$

Theorem 4.2. *For* $\Re(s) < -N$,

$$\int_0^1 a^M \zeta_N(s, a)da$$

$$= \begin{cases} 0, & \text{if } M = N-1, \\ \displaystyle\sum_{n=M+1}^{N} \frac{(-1)^{M-n-1} M!\,\Gamma(1-(s+N-1))}{n!\,\Gamma(1-(s+N+n-M-2))}\zeta_N(s+n-M-1), & \text{if } M < N-1, \\ \displaystyle\sum_{n=N}^{M} \frac{(-1)^{M-n} M!\,\Gamma(1-(s+N-1))}{n!\,\Gamma(1-(s+N+n-M-2))}\zeta_N(s+n-M-1), & \text{if } M \geq N. \end{cases}$$

(4.7)

Proof. Denote by $A = 2(-1)^{N-1}(N-1)!\,\Gamma(1-(s+N-1))$. Then for $M \geq N$,

$$\int_0^1 a^M \zeta_N(s, a)da$$

$$= A\int_0^1 a^M \sum_{k=1}^{\infty}\left[(-z_k)^{s-1}e^{(1-a)z_k} + (-\bar{z}_k)^{s-1}e^{(1-a)\bar{z}_k}\right]da \quad (4.8)$$

$$= A\sum_{k=1}^{\infty}\left[(-z_k)^{s-1}\int_0^1 a^M e^{(1-a)z_k}da + (-\bar{z}_k)^{s-1}\int_0^1 a^M e^{(1-a)\bar{z}_k}da\right]. \quad (4.9)$$

To justify the interchange of the sum and the integral, we note that $z_k = x_k + iy_k$ with $x_k \geq 0$ and $y_k \geq 0$. We recall from the work in [4] that $|z_k| \geq \alpha k$ for some positive constant α. Thus,

$$\left|(-z_k)^{s-1}e^{(1-a)z_k}\right| = |z_k|^{\sigma-1}|e^{z_k}|e^{-ax_k}$$

$$\leq |z_k|^{\sigma-1}|T_{N-1}(z_k)|$$

$$\leq \beta|z_k|^{\sigma-1}|z_k|^{N-1},$$

where β is some positive number and $|z_k| > M$ for some positive number M. Consequently,

$$\left|(-z_k)^{s-1}e^{(1-a)z_k}\right| < \alpha\beta k^{\sigma+N-2} := M_k,$$

for all $|z_k| > M$ and $\sigma + N - 2 < 0$. Since

$$\sum_{k=1}^{\infty} M_k = \alpha\beta\zeta(2 - N - \sigma) < \infty,$$

we see that the infinite series is uniformly convergent and hence the interchange is admissible.

But

$$\int_0^1 a^M e^{(1-a)z_k}da = M!\frac{e^{z_k} - T_{N-1}(z_k) - \sum_{n=N}^{M}\frac{z_k^n}{n!}}{z_k^{M+1}} \quad (4.10)$$

$$= -M!\sum_{n=N}^{M}\frac{z_k^{n-M-1}}{n!}, \quad (4.11)$$

since z_k are roots of $e^z - T_{N-1}(z) = 0$. A similar integral formula holds for the roots $\bar{z}_k$. It follows that

$$\int_0^1 a^M \zeta_N(s, a)da$$

$$= A\sum_{k=1}^{\infty}\left\{\sum_{n=N}^{M}(-1)^{M-n}\frac{M!}{n!}\left[(-z_k)^{s+n-M-2}+(-\bar{z}_k)^{s+n-M-2}\right]\right\}$$

$$= \sum_{n=N}^{M}\frac{(-1)^{M-n}M!\,\Gamma(1-(s+N-1))}{n!\,\Gamma(1-(s+N+n-M-2))}\left\{\sum_{k=1}^{\infty}\left[(-z_k)^{s+n-M-2}+(-\bar{z}_k)^{s+n-M-2}\right]\right\}$$

$$= \sum_{n=N}^{M}\frac{(-1)^{M-n}M!\,\Gamma(1-(s+N-1))}{n!\,\Gamma(1-(s+N+n-M-2))}\zeta_N(s+n-M-1). \tag{4.12}$$

The argument for the cases $M = N-1$ and $M < N-1$ are completely analogous. □

5. A Zero Free Region on the Left Half-Plane

In this section, we will prove that $\zeta_2(s, a)$ has no zeros in the left half-plane $\sigma < -3$, except for one zero in the interval S_m for each integer $m \geq 3$, where S_m is given by

$$S_m = [\sigma_{m-1}, \sigma_m]; \quad \sigma_m = \frac{\theta_1 - (1-a)y_1 - m\pi}{\theta_1}, \quad m = 1, 2, 3, \dots.$$

Here $\theta_1 = 1.2978341024$ and $y_1 = 7.461489286$.

In [4], we had established the following facts about the zeros $z_k = x_k + iy_k = r_k e^{i\theta_k}$ of $e^z - 1 - z = 0$. These zeros can be arranged in an increasing order of magnitude and in doing so we will have the argument also in an increasing order. In fact, both sequences $\{\theta_k\}$ and $\{r_k\}$ are increasing with

$$0 < \theta_k < \pi/2 \quad \text{and} \quad r_k > 2\pi k. \tag{5.1}$$

Furthermore, for $R > 1$, if $r_k = |z_k| > R$ and we define

$$A = \sqrt{\frac{R^2}{(1+R)^2} - \frac{1}{R}}, \quad B = \sqrt{\frac{R}{R-1}}, \tag{5.2}$$

then

$$Ae^{x_k} \leq y_k \leq Be^{x_k}.$$

In the same paper (see Lemma 5.2), we also have

$$2\pi\left(k + \frac{1}{8}\right) < y_k < 2\pi\left(k + \frac{1}{4}\right).$$

Combining these last two inequalities, we see that

$$\left(\frac{2\pi}{B}\right)^{(1-a)}\left(k + \frac{1}{8}\right)^{(1-a)} \leq e^{(1-a)x_k} \leq \left(\frac{2\pi}{A}\right)^{(1-a)}\left(k + \frac{1}{4}\right)^{(1-a)}. \tag{5.3}$$

Theorem 5.1. *Let* $s = \sigma + it$. *If* $\sigma < -3$ *and* $|t| > 1$, *then* $\zeta_2(s, a) \neq 0$.

Proof. We use (4.6) (with $N = 2$) and rewrite $\zeta_2(s, a)$ as

$$\zeta_2(s, a) = f(s, a)(1 + g(s, a)), \tag{5.4}$$

where

$$f(s, a) = -2\Gamma(-s)\left[(-z_1)^{s-1}e^{(1-a)z_1} + (-\bar{z}_1)^{s-1}e^{(1-a)\bar{z}_1}\right], \tag{5.5}$$

and

$$g(s, a) = \sum_{k=2}^{\infty}\left[\frac{(-z_k)^{s-1}e^{(1-a)z_k} + (-\bar{z}_k)^{s-1}e^{(1-a)\bar{z}_k}}{(-z_1)^{s-1}e^{(1-a)z_1} + (-\bar{z}_1)^{s-1}e^{(1-a)\bar{z}_1}}\right]. \tag{5.6}$$

We now estimate $|g(s, a)|$. First note that by triangle inequality, we have

$$|g(s, a)| \leq \frac{2}{|\alpha|}\sum_{k=2}^{\infty}\left(\frac{r_k}{r_1}\right)^{(\sigma-1)} e^{(1-a)(x_k - x_1)}, \tag{5.7}$$

where

$$\alpha = 1 + \frac{(-\bar{z}_1)^{s-1} e^{(1-a)\bar{z}_1}}{(-z_1)^{s-1} e^{(1-a)z_1}}. \tag{5.8}$$

For $t > 1$, we have by the reverse triangle inequality

$$|\alpha| \geq 1 - \left|\frac{(-\bar{z}_1)^{s-1} e^{(1-a)\bar{z}_1}}{(-z_1)^{s-1} e^{(1-a)z_1}}\right| = 1 - e^{-2\theta_1 t} \geq 1 - e^{-2\theta_1} \approx 0.925404,$$

while for $t < -1$, we have

$$|\alpha| \geq \left|\frac{(-\bar{z}_1)^{s-1} e^{(1-a)\bar{z}_1}}{(-z_1)^{s-1} e^{(1-a)z_1}}\right| - 1 = e^{-2\theta_1 t} - 1 \geq e^{2\theta_1} - 1 \approx 12.045544.$$

Thus for $|t| > 1$, we see that $1/|\alpha| \leq 1.0806091$. Using this bound for $1/|\alpha|$ and using (5.3) in (5.7), we get

$$|g(s, a)| \leq 2(1.0806091)\left(\frac{8B}{9A}\right)^{(1-a)} \sum_{k=2}^{\infty} \left(\frac{r_k}{r_1}\right)^{(\sigma-1)} \left(k + \frac{1}{4}\right)^{(1-a)}. \tag{5.9}$$

We use $r_k > 2\pi k$ from (5.1), to estimate the above inequality as

$$|g(s, a)| < 2.1612183\left(\frac{8B}{9A}\right)^{(1-a)} \left(\frac{r_1}{2\pi}\right)^{(1-\sigma)} \sum_{k=2}^{\infty} \frac{1}{k^{1-\sigma}} \left(k + \frac{1}{4}\right)^{(1-a)}. \tag{5.10}$$

We now use the fact that $r_1 \approx 7.48360311$ and choose $R = 10$ in (5.2), to get $A \approx 0.852318$, $B \approx 1.11111$. With these values, the function

$$\phi(\sigma, a) := 2.1612183\left(\frac{8B}{9A}\right)^{(1-a)} \left(\frac{r_1}{2\pi}\right)^{(1-\sigma)} \sum_{k=2}^{\infty} \frac{1}{k^{1-\sigma}} \left(k + \frac{1}{4}\right)^{(1-a)}, \tag{5.11}$$

is an increasing function of σ on $(-\infty, -3)$ and a decreasing function of a on $[0, 1]$. Hence,

$$|g(s, a)| \leq \phi(\sigma, a) \leq \phi(-3, 0) < 1,$$

for $\sigma < -3$ and $0 < a < 1$. By the reverse triangle inequality, it follows that

$$|\zeta_2(s, a)| = |f(s, a)||1 + g(s, a)| \geq |f(s, a)|(1 - |g(s, a)|) > 0,$$

since $|f(s, a)| > 0$ in the region of the hypothesis. This proves the theorem.

Theorem 5.2. *If* $\sigma \leq -3$ *and* $|t| \leq 1$, *then* $\zeta_2(s, a)$ *has exactly one root in the interval* $[\sigma_m, \sigma_{m-1}]$, *where*

$$\sigma_m = \frac{\theta_1 - (1-a)y_1 - m\pi}{\pi - \theta_1}, \quad m = 1, 2, 3, \dots. \tag{5.12}$$

Here $\theta_1 = 1.2978341024$ *and* $y_1 = 7.461489286$.

Proof. Let γ_m be the rectangle with vertices $\sigma_m \pm i$ and $\sigma_{m-1} \pm i$. Let $f(s, a)$ and $g(s, a)$ be as in (5.5) and (5.6), respectively. We shall show that

$$|g(s, a)| < 1 \quad \text{on } \gamma_m. \tag{5.13}$$

Note then that, by the definition of f and g ((5.5) and (5.6)), we have

$$|\zeta_2(s, a) - f(s, a)| = |f(s, a)g(s, a)| < |f(s, a)|,$$

and by Rouche's theorem $f(s, a)$ and $\zeta_2(s, a)$ have the number of roots inside γ_m. Clearly, the roots of

$$f(s, a) = -2\Gamma(-s)\left[(-z_1)^{s-1}e^{(1-a)z_1} + (-\bar{z}_1)^{s-1}e^{(1-a)\bar{z}_1}\right],$$

are the roots of $(-z_1)^{s-1}e^{(1-a)z_1} + (-\bar{z}_1)^{s-1}e^{(1-a)\bar{z}_1}$ and the later has exactly one root in the interval $[\sigma_m, \sigma_{m-1}]$, where σ_m is given by (5.12). On the other hand, from (4.6), we observe that $\zeta_2(\bar{s}, a) = \zeta_2(\bar{s}, a)$. Thus, if $s = \sigma + it$ with $|t| \leq 1$ and $\sigma_m \leq \sigma \leq \sigma_{m-1}$ is a root of $\zeta_2(s, a)$, then $\bar{s}$ is also a root in the same region. Hence $s = \bar{s}$ and the theorem follows.

To prove (5.13), we first consider the vertical line $\sigma = \sigma_m$, $|t| \leq 1$, where σ_m is given by (5.12). Using $s = \sigma_m + it$ in (5.8), we have

$$|\alpha| = \left|1 + e^{-2i\theta_1(s-1)-2iy_1(1-a)}\right|$$

$$= \left|1 + e^{-2\pi i m} e^{-2\theta_1 t}\right| = 1 + e^{-2\theta_1 t}$$

$$> 1.074596014.$$

Hence $1/|\alpha| < 0.9305823$. But then (5.10) becomes

$$|g(s, a)| < 1.8611647 \left(\frac{8B}{9A}\right)^{(1-a)} \left(\frac{r_1}{2\pi}\right)^{(1-\sigma)} \sum_{k=2}^{\infty} \frac{1}{k^{1-\sigma}} \left(k + \frac{1}{4}\right)^{(1-a)}. \quad (5.14)$$

It then follows from the proof of Theorem 5.1, that $|g(s, a)| < \phi(\sigma, a) < 1$, where $\phi(\sigma, a)$ is given by (5.11), for $\sigma = \sigma_m \leq \sigma_2 < -4.29524$. Similar argument can be used on the vertical line $\sigma = \sigma_{m-1} = 1 - \frac{\pi}{\pi - \theta_1}(m - 1)$, $|t| \leq 1$.

Next, we consider the top horizontal boundary $t = 1$, $\sigma_m \leq \sigma \leq \sigma_{m-1}$ of γ_m. On this line, we have, again from the proof of Theorem 5.1, $1/|\alpha| \leq 1.0806091$. Once again, we have $|g(s, a)| < \phi(\sigma) < 1$ since $\sigma_m \leq \sigma_2 < -4.29524$. The argument for the lower horizontal boundary is exactly the same and we have proved our theorem.

Theorem 5.3. *Let* $s = \sigma + it$. *If* $\sigma < -4$ *and* $|t| > 1$, *then* $\zeta_3(s, a) \neq 0$.

Proof. As in Theorem 5.1, we also use (4.6) with $N = 3$. Let $f(s, a)$ and $g(s, a)$ be as in (5.5) and (5.6), respectively, with $r_k e^{i\theta_k}$ being roots of $e^x - 1 - x - x^2/2 = 0$. In [4], we have established that the roots of $e^x - 1 - x - x^2/2 = 0$ can be rearranged to satisfying (5.1). Furthermore, we have $r_1 \approx 9.2053499$, $\theta_1 \approx 1.1406576364$, and $\theta_2 \approx 1.2568294158$. Hence for $|t| > 1$, we have $1/|a| < 0.927818$. On the other hand, for $\sigma < -4$, we have

$$\phi(\sigma, a) < 1.$$

As before, it follows that $|g(s, a)| < 1$ for $\sigma \leq -4$ and $|t| > 1$ and hence $\zeta_3(s, a) \neq 0$. This completes the proof of the theorem.

Combining the arguments of the proofs of Theorem 5.2 (with the obvious modifications) and Theorem 5.3, we obtain the following theorem:

Theorem 5.4. *If* $\sigma \leq -3$ *and* $|t| \leq 1$, *then for each* $m = 3, 4, 5, \cdots$, $\zeta_3(s, a)$ *has exactly one root in the interval* $[\sigma_m, \sigma_{m-1}]$, *where* σ_m *is given by* (5.12). (*Here* $\theta_1 \approx 1.1406576364$).

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